Royce, Boolean Rings, and the T-Relation

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1 Introduction

Royce's sustained interest in technical logic is beyond doubt. One of his first publications, appearing while he was still teaching at the University of California at Berkeley, was a *Logic Primer*, and many of the productions of his later career were articles on logic. Indeed, it can well seem that Royce spent at least ten or eleven years working almost exclusively on logic following his enthusiastic attendance at Peirce's 1898 Cambridge Conference Lectures, entitled *Reasoning and the Logic of Things*. For during this period he filled dozens of notebooks with minute explorations of Boolean functions and relations, investigating them mostly by means of four-circle Venn diagrams.

Less obvious than Royce's devotion to logic is exactly how his achievements in mathematical logic are related to his broader philosophical ideas and, indeed, to his philosophical system as a whole. Perhaps ascertaining this relation is not absolutely critical in connection with his writings of that appeared earlier than 1898, for example *The Religious Aspect of Philosophy*; neither, perhaps, is this critical in connection with his work on the philosophy of loyalty and the philosophy of community. Between the time of publication of the *Logic Primer*, and Royce's attendance at Peirce's lectures, Royce seems not to have devoted an excessive amount of effort to technical logic. The question of the relation of Royce's logic to his system of idealism, however, is much more pressing in connection with Royce's major writings subsequent to 1898, especially *The World and the Individual*, but perhaps also *The Problem of Christianity*.

This question, however, cannot be seriously pursued without an exact and detailed understanding of Royce's accomplishments in mathematical logic. The time seems ripe, then, to make Royce's impressive logical achievements accessible to everyone interested in Royce. The present paper is one step in this direction. The paper explores Royce's effort to make "a definite and new introduction of group-theory into [the] realm of the algebra of logic" , as he puts it in his article "An Extension of the Algebra of Logic," (p. 309 in *Royce's Logical Essays: Collected Logical Essays of Josiah Royce*, ed. Daniel S. Robinson (Dubuque, Iowa: Wm. C. Brown Co., 1951; xvi, 447pp.), pp. 293–309. Additionally, the present paper explores what Royce in this article calls "the T-relation." The T-relation is a "tetradic" —i.e. a four-place—Boolean relation that is intimately tied to his main group operation and that, as Royce's logic notebooks abundantly show, Royce spent a considerable amount of pencil, paper, and time investigating. The T-relation is one of the main subjects of "An Extension of the Algebra of Logic."

Even though group theory and the T-Relation are important for Royce's definition of order relations, and so are important for Royce's late philosophy as a whole, the task of the present paper will nevertheless be strictly to address the need for a clear technical presentation of Royce's ideas considered purely as creations in mathematical logic. The paper will argue that in "An Extension of the Algebra of Logic" Royce develops what is in essence the concept of a Boolean Ring, as well as elaborates certain crucial systems of equations on Boolean Rings. Thus Royce succeeds not merely in introducing the notion of a group into the algebra of logic, but also indeed in introducing the even more elaborate algebraic structure of a ring into the algebra of logic. In the present paper Royce's specifications, including the properties of the T-relation, will be proved to be correct. The paper will also show how Royce's work relates to earlier work on a relation similar to the T-relation done by Stanley Jevons, Ernst Schroeder, and Alfred North Whitehead. In its final section, the paper will extend Royce's results to what we might call "general polyadic T-relations," T-relations that have an arbitrary number of places for Boolean variables, rather than just three or four such places. The final section will also take note of a curious fact: Royce seems greatly to underestimate, and to some extent even to misconstrue, the true significance of what his own considerable mathematical creativity has wrought.

2 Sets, Set Operations, and Boolean Algebras

We have to begin somewhere, so it will be assumed here that the reader is familiar at least with the naive (i.e. non-axiomatic) notions of a *set* and of *set membership*. So as to avoid Russell's paradox, we shall assume that we are always given some non-empty set U as a background for all set construction. We shall call U the "universe of discourse." Out of it all sets under discussion are understood to be formed as subsets of it. Let us now introduce some notation.

2.1 Notation

We shall initially use here upper case letters to denote sets and lower case letters to denote individuals that may not be sets. We shall use the symbol '$\in$' to denote membership of an entity (whether a set or an individual that is not a set) in a set. Thus,

$$a \in A$$

will mean that the entity a is a member of set A. We shall use the symbol '$\notin$' to denote non-membership of an entity in a set. Thus,

$$a \notin A$$

will mean that entity a is not a member of set A. We shall use the symbol '=' to denote identity, both for entities and for sets. Thus,

$$a = b$$

will mean that entity a is the same entity as entity b, and

$$A = B$$

will mean that set A is the same set as set B. The notion of identity for sets is given by the so-called "principle of extensionality," which says that a set A and a set B are the same set if and only if they have exactly the same membership. That is to say:

> $A = B$ if and only if we have both that every member of A is a member of B and that every member of B is a member of A.

We shall use the symbol '$\neq$' to denote non-identity, both for entities and for sets. Thus,

$$a \neq b$$

will mean that entity a is not the same entity as entity b, and

$$A \neq B$$

will means that set A is not the same set as set B. Finally, we shall use the symbol '$\emptyset$' to denote the empty set. $\emptyset$ is also called the *null* set. We can express the fact that the null set has no members at all in this way:

> For all entities a, $a \notin \emptyset$.

It can happen that the membership of a set A is completely subsumed in or by the membership of a set B in the sense that every member of A is a member of B. In such cases we say that the set A is a *subset* of the set B or, equivalently, that the set B is a *superset* of the set A. We shall use the symbol '$\subseteq$'to designate the relation of a set's being a subset of a set. Thus,

$$A \subseteq B$$

means that set A is a subset of set B, or, equivalently, that set B is a superset of set A. In detail what $A \subseteq B$ means is that every member of A is a member of B, in other words that for any entity x, if $x \in A$ then $x \in B$. Take careful note that $A \subseteq B$ does not require that B have any members that A does not have. The only crucial feature is that everything that is a member of A is a member of B. This can happen even when it is also the case that every member of B is a member of A. When we have both that $A \subseteq B$ and that $B \subseteq A$, then obviously A and B have exactly the same members and so $A = B$.

If $A \subseteq B$ but B does have members that A does not have, so that $A \neq B$, then A is said to be a *proper* subset of B and B is said to be a *proper* superset of A. For this relation we can use the symbol '$\subset$' and say that

$A \subset B$ if and only if $A \subseteq B$ but it is not the case that $B \subseteq A$.

It is obvious that

$A \subset B$ if and only if both $A \subseteq B$ and $A \neq B$.

It is also obvious that

$A \subset B$ if and only if both $A \subseteq B$ and there is some entity x such that $x \in B$ and $x \notin A$.

Since we also want to allow the possibility of sets whose members are themselves sets—that is the possibility of sets of sets—we must take careful note of the fact that $\in$ and $\subseteq$ have totally different meanings, as also indeed do $\in$ and $\subset$. (Sometimes it happens that the symbol '$\subseteq$' is read "is included in." This reading, however, as common as it is, tends to encourage the confusion of set membership with subsethood. For this reason the reading of the '$\subseteq$' symbol as "is included in" is to be avoided.)

2.2 Set Operations

We often find it useful to have on hand a set of procedures for doing something to or with a set A or to or with a pair of sets A and B, and thereby producing a set C (which may or may not be the set A or the set B again). Such procedures we shall call *set operations*. A set operation for doing something to or with one set only is called a *one-place* or a *unary* set operation. A set operation for doing something to or with a pair of sets A and B in a given order is called a *two-place* or *binary* operation. The operations introduced here are all binary operations.

2.3 Union

The operation of taking the union of a set A and a set B is a binary operation like the $+$ or $-$ operations of arithmetic. Thus, when A and B are sets, the notation $A \cup B$ denotes the union of A and B. $A \cup B$ is the set that contains as members everything that is either a member of A or a member of B or both. The union operation is *idempotent*, that is: for all sets A, $A \cup A = A$. It is also *commutative*, that is: for all sets A and B, $A \cup B = B \cup A$. It is also *associative*, that is: for all sets A, B, and C, $(A \cup B) \cup C = A \cup (B \cup C)$.

2.4 Intersection

As with the operation of taking the union of a set A and a set B, the operation of taking the intersection of a set A and a set B is a binary operation. Thus, when A and B are sets, the notation $A \cap B$ denotes the intersection of A and B. $A \cap B$ is the set that contains as members everything that is both a member of A and a member of B. The intersection operation is *idempotent*, that is: for all sets A, $A \cap A = A$. It is also *commutative*, that is: for all sets A and B, $A \cap B = B \cap A$. It is also *associative*, that is: for all sets A, B, and C, $(A \cap B) \cap C = A \cap (B \cap C)$.

2.5 Relative Complement

The operation of taking the relative complement of a set B with respect to a set A is again a binary operation. If A and B are any sets, then we proceed to take its the complement of B relative to A, written as

$$A - B$$

as follows. $A - B$ is the set whose membership consists of *every* entity that is a member of A but is *not* a member of B. In other words an arbitrary entity x is a member of $A - B$ if and only if $x \in A$ but it is not the case that $x \in B$, that is to say, if and only if $x \in A$ but $x \notin B$.

When we are taking the relative complement of a set A relative to the universe of discourse U, we may write $(-A)$ instead of $U - A$. In this way the relative complement, when understood as relative to U, appears as a unary operation, though in fact it is an adumbration of a binary operation.

It should be clear from the definition that taking the relative complement with respect to the universe of discourse U of the relative complement of any set A with respect to the universe of discourse U gives the same set A again. That is to say:

$$(-(-A)) = A.$$

The relative complement interacts with union and with intersection to give the two so-called *de Morgan's Laws*:

for all sets A and B, $(-(A \cup B)) = (-A) \cap (-B)$

and

for all sets A and B, $(-(A \cap B)) = (-A) \cup (-B)$.

It follows immediately from the definition of the complement of a set A with respect to the universe of discourse U that $(-\emptyset) = U$. For, by definition $(-\emptyset)$ is the set of all $x \in U$ such that $x \notin \emptyset$. But for absolutely any entity $x \in U$, $x \notin \emptyset$. Thus, $(-\emptyset) = U$. Likewise $(-U) = \emptyset$.

2.6 Set Operations and Propositional Operators

Each of the three set operations just discussed has a special relation to one of the truth-functional propositional operators of logic. It is easiest to appreciate this fact when we use the notation

$$\{x \mid P(x)\}$$

to denote the set whose members are exactly the things that have property P. We read this notation: "The set of all x such that $P(x)$.

Thus, the union operation has a special relation with the logical operator called "disjunction," which is also called the 'inclusive-or' operator, and which is written as '$\vee$'. The basis of this relation is that we can use the $\vee$ to define the union of two sets, as follows:

$A \cup B = \{x \mid x \in A \vee x \in B\}$

Likewise, the intersection operation has a special relation with the logical operator called "conjunction," which is written as '$\wedge$'. The basis of this relation is that we can use the $\wedge$ to define the intersection of two sets, as follows:

$A \cap B = \{x \mid x \in A \wedge x \in B\}$

In a similar manner, the relative complement operation has a special relation with the logical operator called "negation," which is written as '$\neg$'. The basis of this relation is that we can use the $\neg$ (together with the $\wedge$) to define the relative complement of a set B with respect to a set A, as follows:

$A - B = \{x \mid x \in A \wedge \neg(x \in B)\}$

2.7 Power Sets and Boolean Algebras

For any set A we can form its so-called "power set," which will be written here as $\wp(A)$. The set $\wp(A)$ is the set of all subsets of the set A. That is to say, by definition:

$\wp(A) = \{X \mid X \subseteq A\}.$

Since for all A we have both that $\emptyset \subseteq A$ and $A \subseteq A$, it is clear that $\emptyset \in \wp(A)$ and $A \in \wp(A)$. If $B \subseteq A$ and $C \subseteq A$, then it is clear that $B \cup C \subseteq A$ and that $B \cap C \subseteq A$, so that $B \cup C \in \wp(A)$ and $B \cap C \in \wp(A)$. Moreover, if $B \subseteq A$, then $U - B \subseteq A$, so that $U - B \in \wp(A)$. Thus for any set A, $\wp(A)$ is closed under the operations of union, intersection, and complementation with respect to the set A.

Ordinarily, a "Boolean algebra" is defined as an algebra $(0, 1; +, \times, -)$, where the domain designation of the algebra means that the domain has two special entities 0 and 1 in it, and where certain axioms are satisfied. But, according to the Stone Representation Theorem of 1936, every Boolean algebra is isomorphic to some Boolean algebra *of sets*. That is to say, without any loss of generality, we can define a Boolean algebra as an algebra $(\wp(U); \cup, \cap, -)$, where U is some universe of discourse, $\cup$ and $\cap$ have their usual meanings, and $-$ designates complementation with respect to U. In this definition, $\wp(U)$ is the domain of the algebra, and $(\cup, \cap, -)$ is the signature of the algebra. It should be clear that in representing any Boolean algebra as an algebra of sets in this way, the place of the number 0 is taken by the $\emptyset$, the place of the number 1 is taken by U, and the Boolean operations $+$, $\times$, and complementation are taken by $\cup$, $\cap$, and $-$, respectively. The representation of a Boolean algebra as the algebra of the set of all subsets of the universe of discourse U, together with the operations of union, intersections, and relative complementation with respect to U, gives us a nice way of developing a Boolean algebra by appealing to the familiar properties of $\cup$, $\cap$, and $-$, rather than to abstract and perhaps unfamiliar axioms of a purely axiomatic system.

2.8 Royce's Algebra

Although Royce presents his introduction of the theory of groups into the algebra of logic as a pure, uninterpreted algebraic calculus, I shall here present it—in the spirit of the Stone Representation Theorem—as as an algebra of sets. (The main purpose in doing things this way is to give the pure calculus a certain concreteness that should facilitate an understanding of the calculus and yet not involve any loss of generality.) The basic set operation Royce relies upon is usually called "symmetric difference." Just as set union is correlated with the *inclusive* 'or,' so symmetric difference is correlated with the *exclusive* 'or.' The exclusive 'or' placed between two propositions P and Q is simply $\neg(P \Leftrightarrow Q)$. In other words it is the negation of the material equivalence $P \Leftrightarrow Q$, which says that P and Q have the same truth value. Thus $\neg(P \Leftrightarrow Q)$ says that it is not the case that P and Q have the same truth value. Given that the two truth values "True" and "False" are all the truth values there are, $\neg(P \Leftrightarrow Q)$ says that P and Q have *opposite* truth values.

Royce writes the basic operation of symmetric difference with the sign $\circ$. This sign, however, will suggest to contemporary readers a kind of multiplication rather than what it actually is, which is a kind of addition. Royce's sign $\circ$ for this operation, therefore, can be a source of confusion. For this reason Royce's operation will here be written with the sign $\oplus$ rather than Royce's sign $\circ$. The point of this choice of sign is to make very plain that we are dealing here with a kind of addition, a kind of addition that is analogous to the addition of taking set unions but that is still substantially different in content from the taking of set unions. In this paper, the operation indicated by $\oplus$ will be referred to as 'addition,' and the result referred to as a 'sum,' where these words will be placed in single quotes to indicate that we are not referring to set union.

Let A and B be sets that are members of the power set of the universe of discourse U, i.e. let $A \in \wp(U)$ and $B \in \wp(U)$. Then by definition the symmetric difference of A and B is given by

$$A \oplus B = (A \cap (-B)) \cup ((-A) \cap B).$$

In other words, for all $x \in U$,

$$x \in (A \oplus B) \Leftrightarrow (x \in A \wedge x \in (-B)) \vee (x \in (-A) \wedge x \in B).$$

One can also say that for all $x \in U$,

$$x \in (A \oplus B) \Leftrightarrow (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B).$$

The basic algebraic properties of $\oplus$, with two crucial differences, are the same as the basic algebraic properties of $\cup$. Thus, like $\cup$, $\oplus$ is *associative*. That is to say: for all $A, B, C \in \wp(U)$, we have that

$$(A \oplus B) \oplus C = A \oplus (B \oplus C).$$

The associativity of $\oplus$ means that in long strings of 'sums' we can simply ignore the parentheses that, if we were writing with absolute strictness, we would need to place around each binary 'sum.' In other words, we can just omit parentheses altogether from long strings of 'sums.'

The proof in detail of the associativity of $\oplus$ will here be omitted owing to the combination of its complete triviality and its excessive tediousness. The proof amounts to using the basic algebraic properties of $\cup$, $\cap$, and relative complement with respect to U to show that each side of the equation is equivalent to

$$[A\cap B\cap C]\cup[A\cap(-B)\cap(-C)]\cup[(-A)\cap B\cap(-C)]\cup[(-A)\cap(-B)\cap C].$$

Similarly, like $\cup$, $\oplus$ is *commutative*. That is to say: for all $A, B \in \wp(U)$, we have that

$$A \oplus B = B \oplus A.$$

The proof of the commutativity of $\oplus$ is trivial. $B\oplus A = (B\cap(-A))\cup((-B)\cap A)$. But since both $\cup$ and $\cap$ are commutative, the right side of this equation is equivalent to $(A \cap (-B)) \cup ((-A) \cap B)$, which by definition is $A \oplus B$.

The fact that $\oplus$ is both commutative and associative means that in long strings of 'sums' we can not only ignore parentheses around pairs of terms; we can also ignore the order in which the terms appear in a 'sum.'

Again, just as $\cap$ distributes over $\cup$, so also $\cap$ *distributes* over $\oplus$. That is to say: for all $A, B, C \in \wp(U)$, we have that

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C).$$

The proof in detail of this distributivity will here also be omitted owing to its triviality and tediousness. The proof amounts to using the algebraic properties of $\cup$, $\cap$, and relative complement with respect to U to show that each side of the equation is equivalent to

$$[A \cap (-B) \cap C] \cup [A \cap B \cap (-C)].$$

One crucial difference between $\cup$ and $\oplus$ is that, unlike the fact that $\cup$ distributes over $\cap$, it is *not* the case that $\oplus$ distributes over $\cap$. That is to say, we do *not* have that for all $A, B, C \in \wp(U)$ there is an equality between

$$A \oplus (B \cap C)$$

and

$$(A \oplus B) \cap (A \oplus C).$$

For it is straightforward (but tedious) to show that

$$A \oplus (B \cap C) = [A \cap (-B)] \cup [A \cap (-C)] \cup [(-A) \cap B \cap C],$$

and that

$$(A \oplus B) \cap (A \oplus C) = [A \cap (-B) \cap (-C)] \cup [(-A) \cap B \cap C].$$

But in general

$$[A \cap (-B)] \cup [A \cap (-C)] \neq [A \cap (-B) \cap (-C)].$$

A second crucial difference between $\cup$ and $\oplus$ is that for all A, $A \cup A = A$, whereas

$$A \oplus A = [A \cap (-A)] \cup [(-A) \cap A] = \emptyset \cup \emptyset = \emptyset.$$

Here is that fact and nine additional general facts about $\oplus$. The first eight, in the way that we have just seen in connection with Fact 1, follow immediately from the definition of $\oplus$. The last two will be given a small commentary. For all $A, B, C \in \wp(U)$, we have that:

1. $A \oplus A = \emptyset$.
2. $A \oplus \emptyset = \emptyset \oplus A = A$.
3. $A \oplus U = U \oplus A = (-A)$.
4. $A \oplus (-A) = (-A) \oplus A = U$.
5. $A \oplus (-B) = (-A) \oplus B = (-(A \oplus B))$.
6. $A \oplus (A \cap B) = A \cap (-B)$.
7. $A \oplus (A \cap (-B)) = A \cap B$.
8. $A \oplus [(-A) \cap B] = A \cup B$.
9. $A \subseteq B$ if and only if $A \cap (B \oplus U)$, and also $A \subseteq B$ if and only if $A \oplus (A \cap B) = \emptyset$.
10. $A \oplus B = \emptyset$ if and only if $A = B$.

The first clause of Fact 9 is implied by the fact that $A \subseteq B$ if and only if $A \cap (-B) = \emptyset$ and Fact 3. The second clause of Fact 9 is implied by the fact that $A \subseteq B$ if and only if $A \cap (-B) = \emptyset$ and Fact 6.

The proof of Fact 10 is as follows. The 'if' condition here is just a restatement Fact 1. To establish the 'only if' condition, assume that $A \oplus B = \emptyset$. Now, by 'adding' the term B to both sides of the equation, we get that $A \oplus B \oplus B = \emptyset \oplus B$. This gives us, from the associativity of $\oplus$ and Fact 1, that $A \oplus \emptyset = \emptyset \oplus B$. Now, by applying Fact 2, we get that $A = B$.

So, indeed, given the "resources" of the universe of discourse U and the operations $\cap$ and $\oplus$, we can define in terms of these resources alone $\emptyset$ (by Fact 1), $\cup$ (by Fact 8), and $\subseteq$ (by Fact 9). In other words, given these resources alone, we can construct the entire apparatus of the Boolean algebra on $\wp(U)$.

The same discovery, made in Russia by Ivan Ivanovich Zhegalkin almost simultaneously with Royce's work and independently of Royce, and developed by Zhegalkin into an algebra identical to Royce's, made Zhegalkin the "father" of the Russian school of logic. His algebra in Russia is known as "Zhegalkin Algebra." It is a somewhat sad and ironic fact that in America no one speaks of "Royce Algebra," or accords Royce's creations in logic much attention. But, to give credit where credit is due, let us begin to speak of "Royce-Zhegalkin" algebra.

3 Abelian Groups and Boolean Rings

The development of groups and of several other basic constructs of abstract algebra were accomplished in the nineteenth century, by a number of mathematical thinkers. Perhaps most prominent for English speakers were Arthur Cayley and Alfred Bray Kempe. With the theory of groups, as developed by Cayley, Kempe, and others, Royce was certainly familiar. In fact, as will now be shown, Royce's algebra not only exhibits the structure of a group; it also exhibits the more elaborate algebraic structure called a "ring." In fact in Royce's algebra we have what is standardly called a "Boolean Ring."

In order to show that the above is so, let us begin with the notion of a "group." A group is a set X, together with a binary operation $+$, such that the following three conditions are satisfied:

1. $+$ is an associative operation;
2. X contains an "identity element" e for $+$, that is to say, an element e such that for all $x \in X$, $x + e = e + x = x$;
3. for every $x \in X$ there is an "inverse element" $y \in X$ with respect to $+$, that is to say, there is an element $y \in X$ such that $x + y = y + x = e$, where e is the identity element for $+$.

It is a trivial matter to show that an identity element is unique in a group, so that we may speak of *the* identity element of the group. Similarly, it is a trivial matter to show that for any $x \in X$ an inverse element of x is unique, so that we may speak of *the* inverse element of any element $x \in X$. In groups whose group operation is written with the plus sign $+$, it is standard practice to write the identity element with respect to this operation as 0, to write the inverse of any element x with respect to this operation as $-x$, and to define an operation of "subtraction" in the group by defining

$$x - y$$

as

$$x + (-y).$$

For this reason, in order not to confuse operations in Royce's algebra, one should take note that, in virtue of Fact 1, every element of Royce's algebra is its own inverse with respect to $\oplus$. Also, in virtue of Fact 4, it is not the case that the relative complement $(-A)$ of a term A is the inverse of A with respect to $\oplus$. If, therefore we wanted to introduce a subtraction operation into Royce's algebra in the usual way, we would have to use a sign other than the minus sign, say $\ominus$. But there would be little point in doing so, since

for all $A \in \wp(U)$, $(\ominus A) = A$.

and

for all $A, B \in \wp(U)$, $A \ominus B = A \oplus (\ominus B) = A \oplus B$.

A group whose group operation is not only associative but also commutative, that is to say, a group X for which we also have that

for all $x, y \in X$, $x + y = y + x$,

is called a commutative group, or (more commonly) an *Abelian* group. Now, we can see that Royce's algebra is an Abelian group. More exactly, we can see that for any universe of discourse U, the set $\wp(U)$, together with the operation $\oplus$, is an Abelian group. For the operation $\oplus$ is a binary associative operation. Moreover, $\emptyset \in \wp(U)$ and (by Fact 2) we have that for any element $A \in \wp(U)$, $A \oplus \emptyset = \emptyset \oplus A = A$, so that $\emptyset$ is the identity element for $\oplus$ in $\wp(U)$. Moreover, (by Fact 1) we have that for any element $A \in \wp(U)$, $A \oplus A = \emptyset$, so that the inverse element with respect to $\oplus$ of any element $A \in \wp(U)$ is A itself. Thus we have a group, and since $\oplus$ is commutative, this group is an Abelian group.

If now we add some additional features to the notion of an Abelian group, we get the notion of a "ring." A ring is a set X, together with two binary operations $+$ and $\cdot$, such that the following four conditions are satisfied:

1. X together with the operation $+$ is an Abelian group;
2. the operation $\cdot$ is an associative operation;
3. X contains an identity element i for $\cdot$, that is to say, an element i such that for all $x \in X$, $x \cdot i = i \cdot x = x$;
4. the operation $\cdot$ is distributive over the operation $+$, both from the left and from the right; that is to say, for every $x, y, z \in X$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, and $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$.

Additionally, a ring whose operation $\cdot$ is commutative, that is to say, a ring X for which we have that

for all $x, y \in X$, $x \cdot y = y \cdot x$,

is called a *commutative* ring.

But now, we can see that Royce's algebra is a commutative ring. More exactly, we can see that for any universe of discourse U, the set $\wp(U)$, together with the operations $\oplus$ and $\cap$, is a commutative ring. We have already seen that $\wp(U)$, together with the operation $\oplus$, is an Abelian group. Furthermore, the operation $\cap$ is a binary associative operation on $\wp(U)$. Moreover, $U \in \wp(U)$ and it is obvious that for any element $A \in \wp(U)$, that is any set $A \subseteq U$, $A \cap U = U \cap A = A$, so that U is the identity element for $\cap$ in $\wp(U)$. Moreover, $\cap$ is distributive over $\oplus$ from the left, as we have seen; and since both $\cap$ and $\oplus$ are commutative, it follows that $\cap$ is distributive over $\oplus$ from the right. From these facts it follows that $\wp(U)$, together with the operations $\oplus$ and $\cap$, is a ring. And since $\cap$ is commutative, it follows that $\wp(U)$, together with the operations $\oplus$ and $\cap$, is a commutative ring.

A commutative ring whose two operations behave in the manner described in the foregoing material is call a *Boolean ring*. Royce's algebra, then, is a Boolean ring.

4 Royce's T-Relation

In "An Extension of the Algebra of Logic," Royce defines what he calls the 'T-relation' (pp. 295–296), a four-place relation that holds of elements A, B, C, and D if and only if the symmetric difference of A and B is identical to the symmetric difference of C and D. That is to say, by definition:

$T(A, B, C, D)$ if and only if $A \oplus B = C \oplus D$.

By a small algebraic maneuver we can convert the necessary and sufficient condition here of $T(A, B, C, D)$ into one that is a bit easier to grasp. Thus, the condition as Royce writes it obviously implies that

$$A \oplus B \oplus C \oplus D = C \oplus D \oplus C \oplus D.$$

In virtue of Fact 1 about the $\oplus$ operation, the right side of the above equation is equal to $\emptyset$, so that we can also say that the necessary and sufficient condition that Royce writes for $T(A, B, C, D)$ implies that

$$A \oplus B \oplus C \oplus D = \emptyset.$$

But now, this latter condition likewise implies the condition that Royce writes, for the latter condition implies that

$$A \oplus B \oplus C \oplus D \oplus C \oplus D = \emptyset \oplus C \oplus D.$$

And in virtue of Fact 1 and Fact 2 the left hand side of this equation is

$$A \oplus B \oplus \emptyset = A \oplus B,$$

while by Fact 2 the right hand side is

$$C \oplus D.$$

In fact, then, the condition

$$A \oplus B = C \oplus D.$$

that Royce writes for $T(A, B, C, D)$ is simply equivalent to the condition

$$A \oplus B \oplus C \oplus D = \emptyset.$$

Thus, we could define the T-relation by

$T(A, B, C, D)$ if and only if $A \oplus B \oplus C \oplus D = \emptyset$.

This latter way of putting the definition, however, makes much clearer than the original way why the T-relation has the properties Royce attributes to it.

For example, as Royce puts the first of his five properties of the T-relation (p. 296), this relation is "totally symmetrical." This means that we can simply interchange the terms in it arbitrarily. And, indeed, we can. For, since $\oplus$ is both associative and commutative, we can simply rearrange the four terms of the relation in any way we choose without altering the truth value of the relation.

The reformulated definition of the T-relation also makes clear the second property Royce attributes to the T-relation. For, by 'adding'— i.e. with $\oplus$— any one of the four terms to both sides of the equation of the condition for the T-relation, we get as a result the 'added' term on the right side of the equation and the 'sum' of the remaining three terms on the left side of the equation. Let us see how this works in just one case: the term D. (The remaining cases are merely small variants involving some added commutations and associations.) Given the condition of the T-relation, we obviously have

$$A \oplus B \oplus C \oplus D \oplus D = \emptyset \oplus D.$$

But by Fact 1 and Fact 2, this yields

$$A \oplus B \oplus C = D.$$

This condition is also equivalent to the original, and to the condition that Royce writes.

Let us now look at the third property Royce attributes to $T(A, B, C, D)$. He says that we may replace any two of the terms by their complements without altering the truth value of the T-relation. In other words, for example,

$$T(A, B, C, D) \text{ if and only if } T((-A), (-B), C, D).$$

But this just means that

$$A \oplus B \oplus C \oplus D = \emptyset \text{ if and only if } (-A) \oplus (-B) \oplus C \oplus D = \emptyset.$$

But now by Fact 3, $(-A) = A \oplus U$ and $(-B) = B \oplus U$. Hence,

$$(-A) \oplus (-B) \oplus C \oplus D = A \oplus U \oplus B \oplus U \oplus C \oplus D,$$

which, by a series of commutations and associations, is equivalent to

$$A \oplus B \oplus C \oplus D \oplus U \oplus U = A \oplus B \oplus C \oplus D \oplus \emptyset = A \oplus B \oplus C \oplus D.$$

Since, therefore,

$$(-A) \oplus (-B) \oplus C \oplus D = A \oplus B \oplus C \oplus D,$$

it is obvious that the left hand term is equal to $\emptyset$ if and only if the right hand term is equal to $\emptyset$. It should be obvious that a similar argument can be given for any pair of terms.

The fourth property Royce attributes to the T-relation he calls "transitivity by pairs." What this means is that for all A, B, C, D, E, F, if both $T(A, B, C, D)$ and $T(A, B, E, F)$, then $T(C, D, E, F)$. In other words, if

$$A \oplus B \oplus C \oplus D = \emptyset,$$

and

$$A \oplus B \oplus E \oplus F = \emptyset,$$

then

$$C \oplus D \oplus E \oplus F = \emptyset.$$

But this is an elementary derivation. We just 'add' the left sides of the two equations together and equate the 'sum' to the 'added' right sides of the two equations. On the left in the 'sum' we have the term $A \oplus B$ appearing twice, so that their 'sum' is $\emptyset$, which by Fact 2 gives us on the left simply the 'sum' of the terms C, D, E, and F. On the right, we have $\emptyset \oplus \emptyset$, which is $\emptyset$. Thus the proof is accomplished.

The fifth property that Royce attributes to the T-relation is just as simple to prove as the fourth one. The property is that if $T(A, B, C, D)$ and if any two of the terms A, B, C, D are equal to each other, then the remaining two terms are equal to each other; and, conversely, it is always the case that for any two terms A and B, equal to each other or not, $T(A, A, B, B)$. The first clause of this fifth property follows because Fact 1 implies that, if $A = B$, then $T(A, B, C, D)$ reduces to $C \oplus D = \emptyset$, so that, by Fact 10, $C = D$. The second clause follows because Fact 1 implies that $A \oplus A \oplus B \oplus B = \emptyset \oplus \emptyset = \emptyset$.

5 The T-relation Generalized

Having discussed so powerfully the introduction of a group structure into the algebra of logic, and having described so elegantly the intriguing properties of the T-relation, Royce very oddly seems not fully to appreciate his own accomplishments. Indeed, in one respect, he seems even to misunderstand their significance.

For Royce thinks that there is something special about the *tetradic* nature of the T-relation. He takes note of a similar but triadic relation discussed by Jevons, Schroeder, and Whitehead, viz.:

$$A \oplus B \oplus C = \emptyset,$$

and of the total symmetry that exists with respect to it, which symmetry is quite similar to the total symmetry of the T-relation itself. He points out, correctly, that this triadic relation is equivalent to $T(A, B, C, \emptyset)$. But now he seems to think that *beyond this mere fact*, the total symmetry of the triadic relation somehow *derives from* that of the tetradic relation. He even suggests that the properties of the binary operation $\oplus$ derive from those of the T-relation. And he seems almost completely to ignore the extreme richness of the group-theoretical—indeed, ring-theoretical—structure of what he has uncovered. Almost tragically, as it were, he thinks that what he has accomplished in analyzing the T-relation is to have explained what he regards as the relative unfruitfulness in mathematics of the algebra of logic. Here is what he says on page 296.

> This, the only group-operation of the classical Boolean algebra, may thus be regarded as deriving its properties from those of the T-relation. The total symmetry of this tetradic relation is responsible for the simplicity of the group in question, and for its comparative unfruitfulness as a source of novelties in the Boolean calculus.

Of course Royce is correct that the relation

$$A \oplus B = \emptyset$$

may be expressed by $T(A, B, \emptyset, \emptyset)$. But in any further sense than this capability of expression implies, it is simply a mistake to say that the properties of $\oplus$ derive from those of the T-relation. In actual point of fact it is really the properties of $\oplus$ that are responsible for the properties *both* of the Whiteheadian triadic relations and of the tetradic T-relation. We saw this fact in the derivations of the preceding section of the present paper.

As for the comparative unfruitfulness of $\oplus$ as a source of novelties in the Boolean calculus, and as for any relative unfruitfulness in mathematics of the Boolean calculus: nothing could be further from the truth—unless all that Royce means is that nothing in the Boolean calculus that can be proved using $\oplus$ cannot be proved without using it. That much is obviously correct, since $\oplus$ is defined in terms of $\cup$, $\cap$, and relative complement with respect to U. But the fascinating structure of mathematical constructions built using $\oplus$ is in itself a matter of almost endless mathematical richness. Even though Royce has uncovered the gateway to this richness, he seems unsuspecting of the treasures to which his gateway would later lead. It would require many volumes just to give the outline of the story.

But in order both to see that there is really nothing especially extraordinary about the tetradic character of the T-relation, and to see that a rich hierarchy of structures can be built using $\oplus$, let us take note that we may introduce relations of an arbitrary number of places that are similar to the T-relation.

For any positive integer n let us define an n-place T-relation

$$T^n(A_1, A_2, ..., A_n)$$

by

$$A_1 \oplus A_2 \oplus ... \oplus A_n = \emptyset.$$

The left hand term here, viz.:

$$A_1 \oplus A_2 \oplus ... \oplus A_n$$

can be calculated, in terms of the operations $\cup$, $\cap$, and relative complement with respect to U, as follows. This complex term is the union of certain simpler but still complex terms, all of the form $A'_1 \cap A'_2 \cap ... \cap A'_n$, where the term A'_i is either the 'positive' term A_i or the 'negative' term $(-A_i)$. In particular, for n being an *even* number, $A_1 \oplus A_2 \oplus ... \oplus A_n$ is the union of all terms $A'_1 \cap A'_2 \cap ... \cap A'_n$ in which there occurs an *odd* number of 'negative' terms A'_i. And for n being an *odd* number, $A_1 \oplus A_2 \oplus ... \oplus A_n$ is the union of all terms $A'_1 \cap A'_2 \cap ... \cap A'_n$ in which there occurs an *even* number of 'negative' terms A'_i. (Note that 0 is an even number.) In every case of n the number of terms $A'_1 \cap A'_2 \cap ... \cap A'_n$ that are included in the union in question is 2^{n-1}.

Now *all* the relations $T^n(A_1, A_2, ..., A_n)$, and not just Royce's four-place T-relation $T^4(A_1, A_2, A_3, A_n)$, have the properties that Royce attributes to the

T-relation. First, they are all totally symmetrical, owing to the associativity and commutativity of $\oplus$.

Second, given any $n-1$ of the terms of $T^n(A_1, A_2, ..., A_n)$, and the truth of this relation, the remaining term is fully determined as the 'sum' of the $n-1$ given terms.

Third, the truth of the relation is unchanged if any two of the terms are complemented with respect to U.

Fourth, we still get 'transitivity by pairs,' though in a much more general form. Namely, if we have both that $T^n(A_1, A_2, ..., A_n)$ and $T^m(B_1, B_2, ..., B_m)$, and if q of the terms A_i are identical to q of the terms B_j, then the relation $T^{n+m-2q}(C_1, C_2, ..., C_{n+m-2q})$ holds, where $C_1, C_2, ..., C_{n+m-2q}$ are all the A_i that are not identical to any B_j and all the B_j that are not identical to any A_i.

Fifth, if $T^n(A_1, A_2, ..., A_n)$ and n is an even number, and if $A_1 = A_2$, $A_3 = A_4$,..., and $A_{n-3} = A_{n-2}$, then $A_{n-1} = A_n$. Conversely, if n is an even number and $A_1 = A_2$, $A_3 = A_4$,..., $A_{n-3} = A_{n-2}$, and $A_{n-1} = A_n$, then $T^n(A_1, A_2, ..., A_n)$. (We also might point out here that if $T^n(A_1, A_2, ..., A_n)$ and n is an odd number, and if $A_1 = A_2$, $A_3 = A_4$,..., and $A_{n-2} = A_{n-1}$, then $A_n = \emptyset$.

6 Conclusion

Royce's discussion of the introduction of a group structure into the algebra of logic is rich in itself and indeed much richer in potential than Royce himself seems to have realized. The use of the operation $\oplus$ by means of which symmetric difference is defined, allows us to place not merely a group structure on the algebra of logic, but also the structure of a commutative ring. This structure allows for a fascinating mathematical richness, part of which Royce himself uncovered in his T-relation. Contrary to what Royce seems to have thought, there is nothing particularly special about the tetradic nature of the T-relation. But what is of very special interest in the T-relation is that it is a gateway to a unlimited hierarchy of similar relations. Ironically, Royce seems both to have achieved very significant results and yet not clearly to have realized how significant these results were.